Periodic Motion

Periodic Motion

Periodic Motion **is motion that repeats itself over and over.**

The amplitude *A* of the motion is the maximum displacement from the equilibrium position.

The period *T* is the time for one cycle of motion.

The frequency *f* is the number of cycles in a unit of time.

The angular frequency ω is 2π times the frequency.

$$f = \frac{1}{T} \qquad \omega = 2\pi f = \frac{2\pi}{T} \qquad \boxed{T = \frac{2\pi}{\omega} = \frac{1}{f}}$$

Simple Harmonic Motion

Simple Harmonic Motion **is periodic motion in which the restoring force is directly proportional to the displacement from the equilibrium position.**

$$F = -kx = ma$$

$$a = \frac{d^2x}{dt^2} = -\frac{k}{m}x = -\omega^2 x$$

A solution to this differential equation is:

$$x = A\cos(\omega t) \quad \text{where} \quad \omega = \sqrt{\frac{k}{m}}$$

$$\boxed{x = x_{\max}\cos(\omega t + \phi)}$$

Simple Harmonic Motion

The velocity and acceleration of the object are:

$$v = \frac{dx}{dt} = -\omega A\sin(\omega t)$$

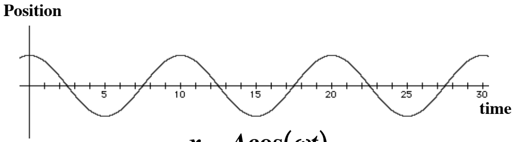
$$a = \frac{d^2x}{dt^2} = -\omega^2 A\cos(\omega t)$$

$$= -\omega^2 x$$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \quad \text{where} \quad \omega = \sqrt{\frac{k}{m}}$$

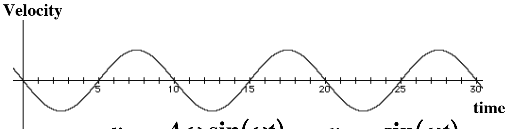
Graphs for Periodic Motion

Position



$$x = A\cos(\omega t)$$

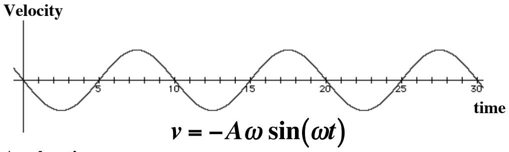
Velocity



$$v = -A\omega\sin(\omega t) = -v_{max}\sin(\omega t)$$

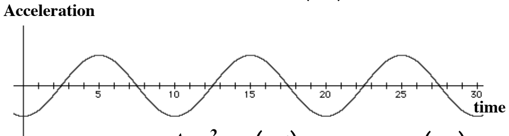
Graphs for Periodic Motion

Velocity



$$v = -A\omega\sin(\omega t)$$

Acceleration



$$a = -A\omega^2\cos(\omega t) = -a_{max}\cos(\omega t)$$

Energy in Simple Harmonic Motion

The total energy is the sum of the kinetic and potential energies of the object.

$$E = K + U = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2$$

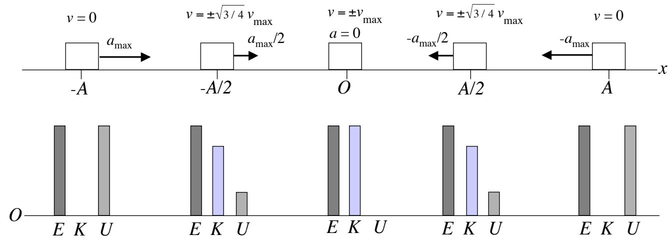
$$E = \tfrac{1}{2}m\left(-\omega A\sin(\omega t)\right)^2 + \tfrac{1}{2}k\left(A\cos(\omega t)\right)^2$$

$$E = \tfrac{1}{2}kA^2\sin^2(\omega t) + \tfrac{1}{2}kA^2\cos^2(\omega t)$$

$$E = \tfrac{1}{2}kA^2\left(\sin^2(\omega t) + \cos^2(\omega t)\right)$$

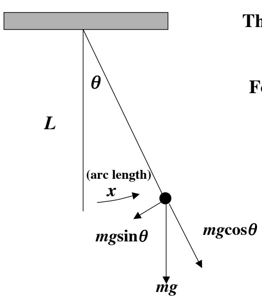
$$\boxed{E = \tfrac{1}{2}kA^2}$$

Energy in Periodic Motion



The Simple Pendulum

A *simple pendulum* is an idealized model consisting of a point mass suspended by a massless, unstretchable string.



The restoring force F is:

$$F = -mg\sin\theta$$

For small displacements $\sin\theta \approx \theta$ and

$$F = -mg\theta$$

$$F = -mg\theta = -mg\frac{x}{L}$$

$$F = -\frac{mg}{L}x$$

The Simple Pendulum

The angular frequency ω of a simple pendulum with small amplitude is:

$$F = -\frac{mg}{L}x = -kx$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{mg/L}{m}} = \sqrt{\frac{g}{L}}$$

The Physical Pendulum

If a pendulum bob cannot be approximated as a point mass, we cannot treat the system as a simple pendulum.

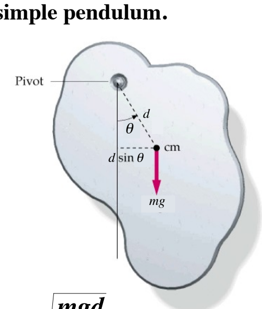
$$\tau = I\alpha$$

$$-mgd\sin\theta = I\frac{d^2\theta}{dt^2}$$

$$-mgd\theta = I\frac{d^2\theta}{dt^2} \quad (\theta \text{ small})$$

$$\frac{d^2\theta}{dt^2} = -\left(\frac{mgd}{I}\right)\theta$$

$$= -\omega^2\theta \quad \text{(SHM)} \qquad \omega = \sqrt{\frac{mgd}{I}}$$



The Torsional Pendulum

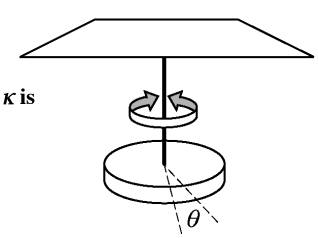
A torsional pendulum consists of a body suspended by a wire attached to a fixed point.

$$\tau = -\kappa\theta = I\frac{d^2\theta}{dt^2}$$

The constant of proportionality κ is called the torsion constant.

$$\frac{d^2\theta}{dt^2} = -\frac{\kappa}{I}\theta = -\omega^2\theta$$

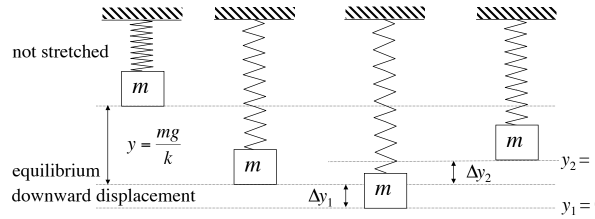
$$\omega = \sqrt{\frac{\kappa}{I}}$$



Simple Harmonic Motion

Springs	Simple Pendulum	Physical Pendulum	Torsional Pendulum
$\omega = \sqrt{\dfrac{k}{m}}$	$\omega = \sqrt{\dfrac{g}{L}}$	$\omega = \sqrt{\dfrac{mgd}{I}}$	$\omega = \sqrt{\dfrac{\kappa}{I}}$
$T = 2\pi\sqrt{\dfrac{m}{k}}$	$T = 2\pi\sqrt{\dfrac{L}{g}}$	$T = 2\pi\sqrt{\dfrac{I}{mgd}}$	$T = 2\pi\sqrt{\dfrac{I}{\kappa}}$
$\boxed{T_S = 2\pi\sqrt{\dfrac{m}{k}}}$	$\boxed{T_P = 2\pi\sqrt{\dfrac{\ell}{g}}}$		

Vertical Spring Amplitude Situation 1:



$$K_1 + U_{e_1} + U_{g_1} = K_2 + U_{e_2} + U_{g_2}$$

$$U_{e_1} = U_{e_2} + U_{g_2}$$

$$\frac{1}{2}k(y + \Delta y_1)^2 = \frac{1}{2}k(y - \Delta y_2)^2 + mg(\Delta y_1 + \Delta y_2)$$

Vertical Spring Amplitude Situation 1:

$$y^2 + 2y\Delta y_1 + \Delta y_1^2 = y^2 - 2y\Delta y_2 + \Delta y_2^2 + \frac{2mg}{k}(\Delta y_1 + \Delta y_2)$$

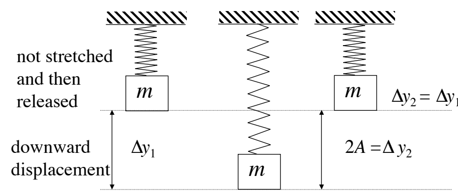
$$2y\Delta y_1 + \Delta y_1^2 = -2y\Delta y_2 + \Delta y_2^2 + \frac{2mg}{k}\Delta y_1 + \frac{2mg}{k}\Delta y_2$$

$$2\frac{mg}{k}\Delta y_1 + \Delta y_1^2 = -2\frac{mg}{k}\Delta y_2 + \Delta y_2^2 + \frac{2mg}{k}\Delta y_1 + \frac{2mg}{k}\Delta y_2$$

$$\Delta y_1^2 = \Delta y_2^2$$

$$\Delta y_1 = \Delta y_2 \quad \text{and} \quad \boxed{A = \Delta y_1}$$

Vertical Spring Amplitude Situation 2:



$$K_1 + U_{e_1} + U_{g_1} = K_2 + U_{e_2} + U_{g_2}$$

$$U_{g_1} = U_{e_2}$$

$$mg\Delta y_1 = \frac{1}{2}k\Delta y_1^2$$

Vertical Spring Amplitude Situation 2:

$$mg\Delta y_1 = \frac{1}{2}k\Delta y_1^2$$

$$\frac{2mg}{k} = \Delta y_1$$

$$\Delta y_1 = 2A$$

$$\boxed{A = \frac{mg}{k}}$$

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Physical Pendulum 1

Torsional Pendulum 1