Rotation of Rigid Bodies

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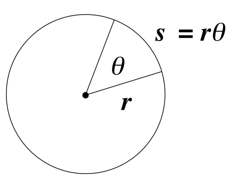
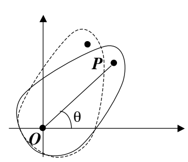
- A *rigid body* has a definite size and shape.
- Forces that act on them do not cause deformations such as stretching, twisting, and squeezing.
- The motions cannot be described as a moving point. Each involves a body that rotates about an axis that is stationary in some inertial frame of reference.

Angle Measurement

For rotational motion, the most natural way to measure angles is radians. The value of θ (in radians) is equal to the arc length s divided by the radius r. The coordinate θ specifies the rotational position of a rigid body at a given instant.





Angular Velocity

The *average angular velocity* is

$$\omega_{av} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

The *instantaneous angular velocity* is

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

At any instant, every part of a rotating rigid body has the same angular velocity moving through different distances.

Angular Acceleration

The *average angular acceleration* is

$$\alpha_{av} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$

The *instantaneous angular acceleration* is

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

At any instant, every part of a rotating rigid body has the same angular acceleration.

Constant Angular Acceleration

The *angular acceleration* is

$$\alpha = \frac{\omega - \omega_o}{t - 0} \quad \text{or} \quad \boxed{\omega = \alpha t + \omega_o}$$

The *average angular velocity* is

$$\omega_{av} = \frac{\omega_o + \omega}{2}$$

Using the relationship between angular velocity and total angular displacement the above equation becomes

$$\theta = \tfrac{1}{2}(\omega + \omega_o)t + \theta_o$$

Constant Angular Acceleration

The equations used to describe linear motion all have rotational equivalents.

$$\boxed{\omega = \alpha t + \omega_o}$$

$$\theta = \tfrac{1}{2}(\omega + \omega_o)t + \theta_o$$

$$\boxed{\theta = \tfrac{1}{2}\alpha t^2 + \omega_o t + \theta_o}$$

$$\omega^2 = \omega_o^2 + 2\alpha(\theta - \theta_o)$$

Relating Linear and Angular Kinematics

- **When a rigid body rotates about a fixed axis, every particle in the body moves in a circular path.**
- **The speed of a particle is directly proportional to its angular velocity.**
- **The distance a particle moves is the arc length *s* which is related to the angle and radius of the circle.**

$$s = r\theta \qquad \frac{ds}{dt} = r\frac{d\theta}{dt} \qquad \boxed{v = r\omega}$$

Relating Linear and Angular Kinematics

The acceleration of a particle moving in a circular path has both tangential and centripetal components

$$a_{tan} = \frac{dv}{dt} = \frac{d(r\omega)}{dt} = r\frac{d\omega}{dt} = r\alpha$$

This component is always tangent to the circular path of the particle.

$$\boxed{a_{rad} = \frac{v^2}{r} = \omega^2 r}$$

This component always points towards the axis of rotation.

Energy in Rotational Motion

- **All the particles in a rotating rigid body have mass and are in motion, so they have kinetic energy.**
- **The kinetic energy of the i^{th} particle can be expressed as**

$$\tfrac{1}{2}m_i v_i^2 = \tfrac{1}{2}m_i r_i^2\omega^2$$

- **The total kinetic energy of the body is the sum of the kinetic energies of all its particles**

$$K = \sum_i \tfrac{1}{2}m_i r_i^2\omega^2 = \tfrac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2$$

Moment of Inertia

The sum of the products of the mass of each particle times the square of their distances from the axis of rotation is called the *moment of inertia* of the body.

$$I = \sum_i m_i r_i^2$$

In terms of moment of inertia *I*, the *rotational kinetic energy* of a rigid body is

$$\boxed{K = \tfrac{1}{2}I\omega^2}$$

Parallel-Axis Theorem

$$I_p = I_{cm} + Md^2$$

I_{cm} - the moment of inertia of mass *M* about an axis through its center of mass.

I_p - the moment of inertia about an axis parallel to one through its center of mass but displaced from it by a distance *d*.

Therefore, a rigid body has a lower moment of inertia about its center of mass than about any other parallel axis.

Moment of Inertia Calculations

When a rigid body cannot be represented by a few point masses but is a continuous distribution of mass the moment of inertia is represented by

$$\boxed{I = \int r^2 dm = \sum mr^2}$$

r **= distance of *dm* from axis of rotation**

dm **= differential mass element**

For a body with a density ρ, *dm* can be written in *terms of density and volume* and the moment inertia is

$$I = \int r^2 \rho dV$$

Dynamics of Rotational Motion

Torque

The quantitative measure of the tendency of a force to cause or change the rotational motion of a body is called torque τ.

In terms of vectors:

$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}} = rF\sin\phi$$

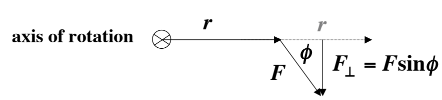
F **= force**

r **= the distance between the force and the axis of rotation**

ϕ = the angle between *r* and *F*

Torque

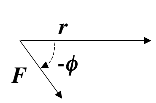
Only the perpendicular component of the force *F* relative to *r* contributes to torque.

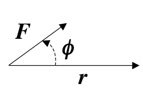


$$\tau = rF\sin\phi$$

Torque

The direction of the torque is either clockwise (-) or counterclockwise (+).





Rotational Analog of Newton's Second Law

The net torque on a rigid body equals the body's moment of inertia about the rotation axis times its angular acceleration.

$$\boxed{\sum \tau = I\alpha}$$

$$\boxed{\vec{\alpha} = \frac{\sum \vec{\tau}}{I} = \frac{\vec{\tau}_{net}}{I}}$$

Kinetic Energy of Rotating Bodies

The motion of a body can always be divided into *independent* translation of the center of mass and rotation about the center of mass.

$$KE = \tfrac{1}{2} M v_{cm}^2 + \tfrac{1}{2} I \omega_{cm}^2$$

(translation) (rotation)

When an body is *rolling without slipping*

$$v_{cm} = R\omega$$

Work in Rotational Motion

The total work W done by the torque during an angular displacement from θ_1 to θ_2 is

$$W = \int_{\theta_1}^{\theta_2} \tau \cdot d\theta$$

If the torque is constant while the angle changes by a finite amount $\Delta\theta$

$$W = \tau \cdot \Delta\theta$$

Work in Rotational Motion

When a torque does work on a rotating rigid body, the kinetic energy changes by an amount equal to the work done.

$$\tau \cdot d\theta = I\alpha \cdot d\theta = I\frac{d\omega}{dt} d\theta = I\frac{d\theta}{dt} d\omega = I\omega \cdot d\omega$$

$$W_{tot} = \int_{\omega_1}^{\omega_2} I\omega \cdot d\omega = \tfrac{1}{2} I\omega_2^2 - \tfrac{1}{2} I\omega_1^2$$

Work-Energy Theorem

The change in rotational kinetic energy of a rigid body equals the work done by the forces exerted from outside the body.
This is analogous to the work-energy theorem for a particle.

$$W_{tot} = \Delta KE = \tfrac{1}{2} I\omega_2^2 - \tfrac{1}{2} I\omega_1^2$$

Power in Rotational Motion

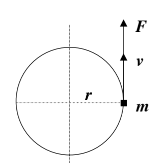
Power is simply the work divided by the time interval during which the angular displacement occurs

$$\frac{dW}{dt} = \tau \frac{d\theta}{dt}$$

$$\boxed{P = \tau\omega}$$

Angular Momentum

Consider a small point mass m at a distance r from the axis of rotation, moving with velocity v and acted upon by a tangential force F.

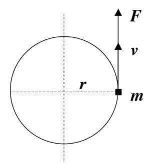


Then by Newton's Second Law

$$F = \frac{dp}{dt} = \frac{d(mv)}{dt}$$

Angular Momentum

If we multiply both sides of this equation by *r* we get



$$Fr = r\frac{d(mv)}{dt} = \frac{d(rmv)}{dt} = \tau$$

So torque is the rate of change of the quantity (*rmv*) which is called angular momentum *L*.

$$\boxed{L = rmv}$$

Angular Momentum

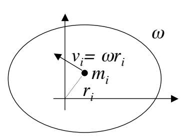
The analog of momentum *p* of a particle is angular momentum *L*.

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} = mvr\sin\phi$$

The rate of change of angular momentum of a particle equals the torque of the net force acting on it.

$$\frac{dL}{dt} = \vec{r} \times \vec{F} = \vec{\tau}$$

Total Angular Momentum for a Rigid Body



$$L_i = m_i v_i r_i$$

$$L_i = m_i(r_i\omega)r_i = m_i r_i^2\omega$$

The total angular momentum of body is the sum of all the particles on the body.

$$L = \sum_i L_i = \left(\sum_i m_i r_i^2\right)\omega$$

$$\boxed{L = I\omega} \qquad \boxed{\vec{L} = \vec{r} \times \vec{p} = I\vec{\omega}}$$

Conservation of Angular Momentum

When the net external torque acting on a system is zero, the total angular momentum of the system is constant.

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{net}$$

So if the net torque is zero, then *L* is constant.

$$\sum L_i = \sum L_f$$

Equilibrium

First condition for equilibrium (no acceleration)

$$\sum F_x = 0,\ \sum F_y = 0,\ \sum F_z = 0,$$

Second condition for equilibrium (no rotation)

$$\sum \tau = 0$$

The sum of the torques due to all external forces acting on the body, with respect to any specified point, must be zero.